Last Time

PP(λ) is a counting process satisfying 3 basic props:

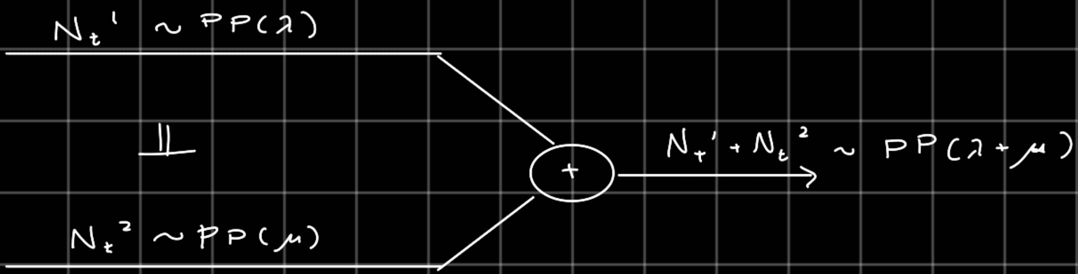
1. Starts at 0 ($N_0 = 0$)
2. Has stationary increments ($N_{t+s} - N_s \sim \text{Poisson}(t\lambda)$)
3. $(N_t)_{t \geq 0}$ has independent increments

Properties

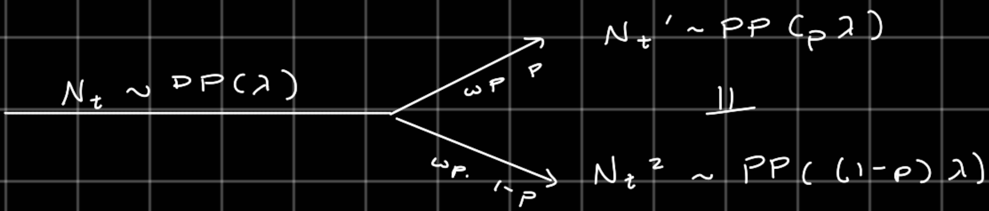
1. Conditioning on arrivals:

$$(T_1 \ldots T_n) \mid \{N_t = n\} \overset{d}{=} \text{order statistics of } n \text{ iid Unif}(0, t) \text{ rvs}$$

2. Adding 2 indep. PP's results in a new PP



3. Splitting up a PP results in indep PP's



Exam

Example: Suppose cars/trucks pass tollbooth according to indep PPs of rates λ_c & λ_T respectively

Q1/ At time t_0 what's the probability that the next 3 vehicles are all cars?

A₁/ Think of merging & splitting

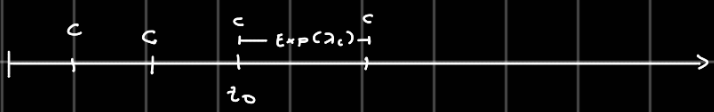
$$\left(\frac{\lambda_c}{\lambda_c+\lambda_T}\right)^3 \quad . \quad \text{why?}$$

↳ consider combined vehicle process which is $PP(\lambda_c+\lambda_T)$. I can obtain car/truck process from this by marking arrivals with "C"/"T" wp $\frac{\lambda_c}{\lambda_c+\lambda_T}$, $\frac{\lambda_T}{\lambda_c+\lambda_T}$ respectively

Key idea:
identify structure of problem & exploit properties of Poisson processes

Q₂/ Suppose 2 cars have passed & 3 trucks have passed at time t_0. What is the expected time at which 10^{th} car passes?

A₂/ use exponentials



$$T_{10} = t_0 + \sum_{i=1}^{8} \tau_i \qquad \tau_i \overset{iid}{\sim} \exp(\lambda_c)$$

$$\mathbb{E}[T_{10}] = t_0 + \frac{8}{\lambda_c}$$

Q₃/ At time T_{10} (10 cars have passed), what's the expected # of trucks we will have seen?

A₃/ Let N = # of trucks btwn time t_0 & T_{10}.

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N \mid T_{10}]] = \mathbb{E}[\lambda_T(T_{10}-t_0)] = \lambda_T\left(t_0+\frac{8}{\lambda_c}\right)-\lambda_T t_0$$

$$N \mid T_{10} \sim \text{Poisson}(\lambda_T(T_{10}-t_0)) \qquad = \frac{8\lambda_T}{\lambda_c}$$

$$\rightarrow \quad 3 + \frac{8\lambda_T}{\lambda_c}$$

Random Incidence Paradox

Let $(N_t) \sim PP(\lambda)$. Suppose I pick a time t_0 far into the future.

Q/ What's expected time btwn previous & next arrival?

A/ $\frac{2}{\lambda}$ → intuitively it's not $\frac{1}{\lambda}$ bc if you choose t_0 at random, t_0 is more likely to land in a long interval than a short interval

Idea: Suppose t_0 falls in $[T_i, T_{i+1})$

$$T_{i+1} - t_0 \sim Exp(\lambda) \text{ by memoryless property}$$

For "reverse" interval: $P(t_0 - T_i > s) = P(N_{t_0} - N_{t_0 - s} = 0)$

$$= e^{-\lambda s}$$

for $s \leq t_0$

⇒ if t_0 very large ($t_0 >>> 0$)

⇒ $t_0 - T_i \sim Exp(\lambda)$

$$\Rightarrow \mathbb{E}[T_{i+1} - T_i] = \mathbb{E}[T_{i+1} - t_0] + \mathbb{E}[t_0 - T_i]$$

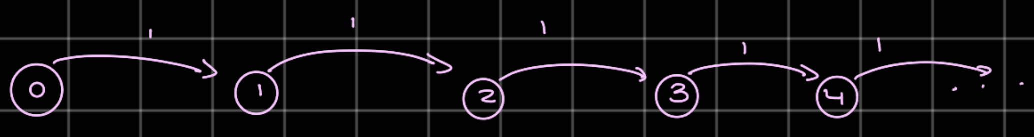
$$= \frac{1}{\lambda} + \frac{1}{\lambda}$$

$$= \frac{2}{\lambda}$$

Continuous Time Markov Chains (CTMC)

Example

Let's think about Poisson process as follows:



} this captures the counting process of PP's but not the timing...

start at state 0, then "hold" in each state for $Exp(\lambda)$ time before next transition

If $(X_t)_{t \geq 0}$ is a process where X_t = state at time t, then $(X_t) \sim PP(\lambda)$.

This is an example of a CTMC. In general: CTMC = DTMC + exponential holding times

Recall: DTMC was characterized by P-matrix

↳ Analogy for CTMCs is so-called generator matrix (aka rate matrix / Q-matrix), denoted by Q.

- Q matrix satisfies the 3 properties:

① $[Q]_{ij} \geq 0 \quad \forall i \neq j, \; i,j \in S$ (off diagonal elements of Q are non-negative)

② $\sum_j [Q]_{ij} = 0 \quad \forall i \in S$ (each row of Q sums to zero)

Equivalently: $[Q]_{ii} = -\sum_{i \neq j} [Q]_{ij}$ ⊛

Define: transition rate for state i as $q_{ii} := -[Q]_{ii}$

↳ tells you exact distribution for holding state i

If we divide ⊛ by q_i, then we have

$$[Q]_{ij} = q_i \frac{[Q]_{ij}}{q_i} = q_i P_{ij} \qquad i \neq j$$

for some $(P_{ij})_{i,j} \in S$

Note: $P_{ij} = \dfrac{[Q]_{ij}}{q_i} \geq 0$

$\sum_{j \neq i} P_{ij} = \dfrac{\sum_{j \neq i} [Q]_{ij}}{-[Q]_{ii}} = 1$

these P_{ij}'s are called transition probabilities for "embedded chain" / "jump chain".

$$= P = \begin{bmatrix} 0 & P_{01} & P_{02} & \cdots \\ P_{10} & 0 & P_{12} & \cdots \\ P_{20} & & \ddots & \\ \vdots & & & 0 \end{bmatrix}$$

← this is a valid transition prob matrix. It defines the "embedded chain"

A CTMC w/ rate matrix Q works as follows:

1) Set initial state $X_0 = i$

2) Hold in state i for $Exp(q_i)$ time, then jump to next state j with prob P_{ij}

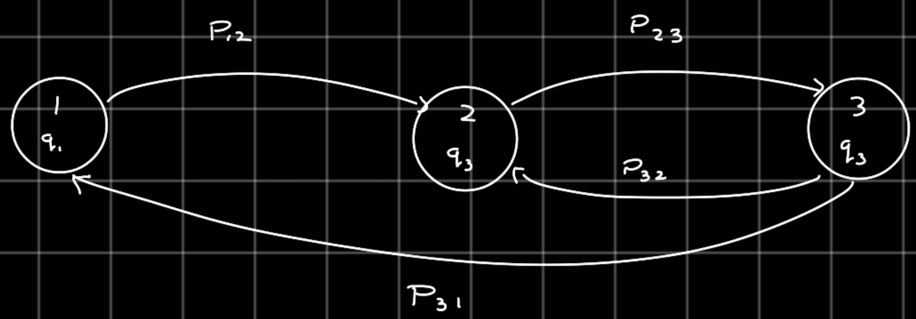
3) Hold in next state j for $Exp(q_j)$ time then

jump to next state with prob P_{jk}.

4) : CTMC is process $(X_t)_{t\geq 0}$ where X_t = state at time t.

Note : No self-loops in the embedded chain bc you can think of staying in the current posn for a certain amt of time as a sort of "self loop"

Picture to have in mind:



$$Q = \begin{bmatrix} -q_1 & q_1 P_{12} & 0 \\ 0 & -q_2 & q_2 P_{23} \\ q_3 P_{31} & q_3 P_{32} & -q_3 \end{bmatrix}$$

Q/ What do we call this a continuous time MC?

A/ The process $(X_t)_{t\geq 0}$ has Markov property.

$$P(X_{t+\tau} = j \mid X_t = i, X_s = i_s, 0 \leq s \leq t) = P(X_{t+\tau} = j \mid X_t = i)$$
$$= P(X_\tau = j \mid X_0 = i)$$

Examples:

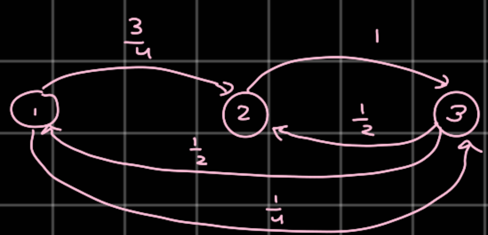
$$Q = \begin{bmatrix} -4 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & -2 \end{bmatrix}$$

Q_1) What are the transition rates?

A_1) $q_1 = 4$ $q_2 = 2$ $q_3 = 2$

Q_2) What's the embedded chain look like?

A_2)



Note: For CTMC's we conventially draw state transition diagrams with arrows labeled by transition rates, eg

A_2 /